Delay and periodicity

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Systems with time delay play an important role in modeling of many physical and biological processes. In this paper we describe generic properties of systems with time delay, which are related to the appearance and stability of periodic solutions. In particular, we show that delay systems generically have families of periodic solutions, which are reappearing for infinitely many delay times. As delay increases, the solution families overlap leading to increasing coexistence of multiple stable as well as unstable solutions. We also consider stability issue of periodic solutions with large delay by explaining asymptotic properties of the spectrum of characteristic multipliers. We show that the spectrum of multipliers can be split into two parts: pseudocontinuous and strongly unstable. The pseudocontinuous part of the spectrum mediates destabilization of periodic solutions.

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I. INTRODUCTION

The dynamical behavior of various physical and biological systems under the influence of delayed feedback or coupling can be modeled by including terms with delayed arguments in the equations of motion. When the delay becomes longer than the other characteristic time scales of the system, a complicated and high-dimensional dynamics can appear [1–6]. The analysis and control of such dynamical regimes is important for many applications including lasers with optical feedback or coupling [5,7,8], neural activity control [9,10], and many others [11]. For instance, the following complicated regimes have been observed in lasers with delayed feedback: low-frequency fluctuations [12], regular pulse packages [13], and coherence collapse, just to mention a few.

One of the fundamental question in the analysis of systems with delay concerns properties of periodic solutions. Periodic solutions of any dynamical system, including also systems with delay, are important parts of the dynamics. When such solutions are stable, they can be directly observed experimentally or numerically. In the case, when such solutions are unstable, they play an important role, e.g., by determining of a set of admissible initial values to be attracted to some stable steady state (basin boundary), or by forming fundamental blocks of a chaotic attractor [14,15]. Finally, unstable as well as stable periodic solutions can play an important role in mediating any kind of soft or hard transitions as some control parameters are varied.

This paper is devoted to the study of generic properties of periodic solutions in systems with a constant time delay,

$$x'(t) = f[x(t), x(t-\tau)], \tag{1}$$

where $x \in R^n$ and $\tau > 0$ is the time delay. Since we investigate the influence of the delay, we assume τ to be our control parameter. Delay has been used as a parameter in various applications: chaotic systems with feedback [16,17], network motifs [18,19], large networks or arrays of oscillators with delayed coupling [20–23], mechanical systems [24–27], laser systems with feedback [28] and coupling [29], coupled neurons [30,31], chemical oscillators [32], and delayed feedback control [8,10,33–40]. We believe that our results are applicable to the all above-mentioned cases as well as to many others, which include time delay as a controllable parameter.

The plan of the paper is as follows. Section II starts by showing that periodic solutions in system (1) are forming branches with respect to the control parameter τ. These branches are reappearing infinitely many times for different delay values. As the delay increases, the solution branches overlap leading to increasing coexistence of multiple stable as well as unstable periodic solutions. The number of coexisting solutions is shown to be linearly proportional to the delay time. Further in Sec. III, we consider stability properties of periodic solutions for systems with large delay by explaining asymptotic properties of the spectrum of their characteristic multipliers. The spectrum of multipliers can be split into two parts: pseudocontinuous and strongly unstable. Such situation is similar to the case of steady states in delay systems with large delay [3,41,42]. The pseudocontinuous spectrum mediates bifurcations of periodic solutions for systems with large delay. Sections IV and V discuss some implications of the existence of pseudocontinuous spectrum and possibility for its numerical computation.

The obtained results provide a better understanding of mechanisms behind the growing multistability and dynamic complexity in systems with delay. In particular, we show that coexistence of multiple stable (as well as unstable) periodic solutions is a natural feature of delay systems. The growing "effective dimension" of dynamics with the growing delay is supported by the fact that the dimensionality of unstable manifolds of periodic solutions also grows linearly with delay. Our results will be illustrated using two systems. The first is the Duffing oscillator with delay,

$$x''(t) + dx'(t) + ax(t) + x^3(t) + b[x(t) - x(t-\tau)] = 0, \tag{2}$$

where $x(t)$ is a real variable, d, a, and b are the positive real parameters corresponding to damping, linear part of stiffness, and delayed feedback strength. The second is the Stuart-Landau oscillator with delayed feedback,

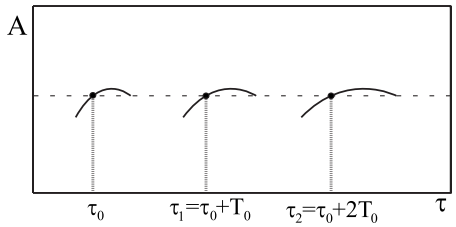

FIG. 1. Reappearance of a periodic solution $x_0(t)$ for the delays $\tau_n=\tau_0+nT_0$, $n=1,2,3,\ldots$. T_0 is the period of $x_0(t)$. The figure is schematic and shows a part of the branch of periodic solutions, which can be obtained locally by continuation of $x_0(t)$ along the parameter τ. τ is time-delay and A is the amplitude of a periodic solution or some of its components.

$$z'(t) = (\alpha + i\beta)z(t) - z(t)|z(t)|^2 + z(t-\tau), \tag{3}$$

where $z(t)$ is a complex variable, α and β are the real parameters.

II. REAPPEARANCE OF PERIODIC SOLUTIONS AND BRANCHES

In this section we show that periodic solutions of system (1) reappear infinitely many times for different and well-specified values of delay τ. We will also show that such solutions typically form branches, which can be mapped one onto another by some similarity transformation.

A. Reappearance of periodic solutions

Let us consider system (1), which possesses a periodic solution $x_0(t)$ for a time delay $\tau=\tau_0$. Let T_0 be the period of this solution. Then it is easy to check that the same periodic solution exists in system (1) with time delay $\tau_1=\tau_0+T_0$. Indeed, substituting $x_0(t)$ into Eq. (1) we obtain

$$\begin{aligned} x_0'(t) &= f[x_0(t),x_0(t-\tau_1)] = f[x_0(t),x_0(t-\tau_0-T_0)] \\ &= f[x_0(t),x_0(t-\tau_0)], \end{aligned}$$

where the periodicity of $x_0(t)$ is taken into account. Similarly, the solution $x_0(t)$ reappears for all values $\tau_n=\tau_0+nT_0$, with $n=1,2,3,\ldots$ of the delay. In Fig. 1 we presented a schematic plot of reappearance of periodic solutions, where A is the amplitude of $x_0(t)$ (maximum) as a function of τ.

B. Reappearance of branches of periodic solutions

A periodic solution $x_0(t)$ can be generically continued to a branch of periodic solutions $x_0(t;\tau)$ with respect to the parameter τ, at least in some neighborhood of τ_0. Denote $T(\tau)$ to be the period of these solutions along the branch. Since each individual periodic solution reappears for every delay time $\tau+nT(\tau)$, the whole branch will appear infinitely many times as well (see Fig. 1). It is naturally to distinguish the first branch with the smallest delay and introduce the notion of *primary branch*, which satisfies $\tau<T(\tau)$. For convenience, let us parametrize the primary branch using the parameter l, which coincides with the delay on this branch $x_0(t;l):=x_0(t;\tau)$. A solution $x_0(t;l)$, which corresponds to some parameter value l, will appear again on the n-th branch at time delay $\tau(n,l)=l+nT(l)$ (see Fig. 1). Thus, we obtain the representation of the nth branch,

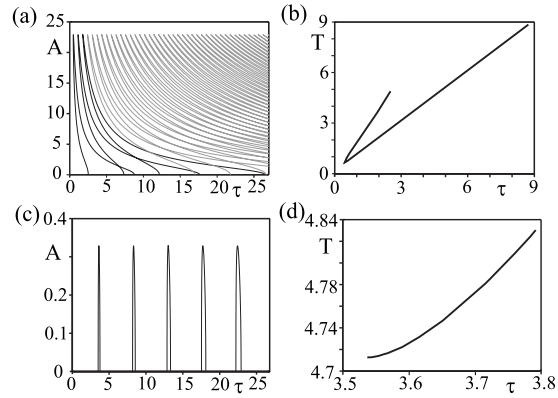

FIG. 2. Branches of periodic solutions for the Duffing oscillator. Left panel shows the amplitude A of the solutions $x_n(t;\tau)$ versus time delay τ, right panel shows dependence of the period T along the primary branch. Parameter values: [(a) and (b)] $a=0.5$, $b=0.6$, and $d=0.06$; [(c) and (d)] $a=1.38$, $b=0.4$, and $d=0.3$. In (a): the first three branches are plotted in black and the rest in gray.

$$x_n[t;\tau(n,l)] = x_n[t;l+nT(l)] = x_0(t;l). \tag{4}$$

The corresponding mapping, which maps delay times, is given as follows:

$$l \to \tau(n,l) = l + nT(l). \tag{5}$$

Examples of the above described branches can be numerously found in the research literature [16,19,20,22,25,32,36,43]. Usually, these branches can be found numerically. A useful tools for finding such branches is the continuation software DDE-BIFTOOL [44]. In fact, only the primary branch should be computed while the others can be obtained using the transformation [Eq. (4)].

For the model of the Duffing oscillator [Eq. (2)], we have found two types of branches, which are presented in Fig. 2. Some other examples are given in Fig. 4. Note that due to the nontrivial dependence of the period $T(\tau)$ along the branch (see the right panel of the figure), the mapping [Eq. (5)] is not just a parallel shift but it also has more complicated properties, which will be studied in Sec. II C.

C. Properties of the branches

As we have seen, the primary branch of periodic solutions $x_0(t;l)$ is characterized by a period function along the branch $T(l)$. It is clear that the other branches have the same period dependence $T(l)$ since they consist of the same solutions. Mapping [Eq. (5)] implies that the function $T(l)$ determines uniquely how branches reappear for larger delay times. Let us discuss main properties of the map [Eq. (5)] and corresponding implications.

1. Stretching and squeezing

Under the transformation [Eq. (5)] some parts of the branch will be stretched and some parts squeezed. In particular if

$$\left|\frac{\partial\tau(n,l)}{\partial l}\right| = |1 + nT'(l)| > 1, \tag{6}$$

then the corresponding part will be locally stretched. Here $T'(l)=dT(l)/dl$ is the derivative of the period function. If the inverse inequality

$$\left|\frac{\partial\tau(n,l)}{\partial l}\right| = |1 + nT'(l)| < 1 \tag{7}$$

is satisfied, the corresponding branch parts will be locally squeezed. With the increasing of n (which is equivalent to increasing τ) almost all parts of the branches will be stretched, since [Eq. (6)] will be satisfied for large enough n. Hence, the branches become eventually wider and occupy larger τ intervals. This leads to the growing overlapping of branches and growing coexistence of periodic solutions with increasing τ. This effect is clearly visible in Figs. 2(a) and 4.

2. Multistability

Let us describe the above-mentioned phenomenon on a more quantitative level. We will show that the number of coexistent periodic solutions grows linearly with τ and give estimations for the corresponding coefficient. Let us consider the two possible cases:

Case 1. The primary branch is confined to some interval of τ as in the case shown in Figs. 2(a)–2(d). This means that the primary branch ranges from $l_{\min}$ until $l_{\max}$ ($l_{\min}$ can be zero).

Case 2. The primary branch is bounded only from below, i.e., it ranges from $l_{\min}$ until $+\infty$; see an example in Fig. 4(a).

Consider the first case. Let us denote $T_{\max}$ and $T_{\min}$ to be the maximum and the minimum of the period function $T(l)$ on the interval $l_{\min}\le l\le l_{\max}$. If $T_{\max}=\infty$ then the problem can be reduced to the case 2 since the next branch $x_1[t;l+T(l)]=x_0(t;l)$ will be stretched up to $\tau=\infty$ by the mapping [Eq. (5)]. Hence, $T_{\max}$ and $T_{\min}$ can be considered to be bounded. In this case, all other branches are also bounded and exist for delay values $\tau(n,l)=l+nT(l)$, $l_{\min}\le l\le l_{\min}$. In particular, the nth branch ranges from

$$\tau_{\min}(n,l) = \min_{l_{\min}\le l\le l_{\max}} \tau(n,l) = \min_{l_{\min}\le l\le l_{\max}} [l + nT(l)]$$

until

$$\tau_{\max}(n,l) = \max_{l_{\min}\le l\le l_{\max}} \tau(n,l) = \max_{l_{\min}\le l\le l_{\max}} [l + nT(l)].$$

For large enough n, the minimal and maximal bounds of the branches can be well approximated as follows:

$$\tau_{\min}(n,l) = n \min\left[\frac{l}{n} + T(l)\right] \approx nT_{\min}, \tag{8}$$

$$\tau_{\max}(n,l) = n \max\left[\frac{l}{n} + T(l)\right] \approx nT_{\max}, \tag{9}$$

up to the terms of order 1. Let us now find how many branches are overlapping for some sufficiently large delay value τ. It is clear that these branches should satisfy the condition

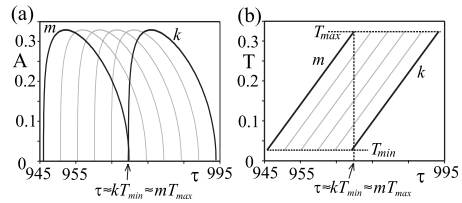

FIG. 3. Illustration to the derivation procedure of the formula for the number of overlapping branches at time delay τ. Left panel shows the amplitude A of the solutions versus time delay τ, right panel shows dependence of the period T along the branch. Shown is the delayed Duffing oscillator with parameter values $a=1.38$, $b=0.4$, $d=0.3$, see also Fig. 2(c). The branches k and m are shown in black. More details are given in text.

$$\tau_{\min}(n,l) < \tau < \tau_{\max}(n,l).$$

Let m be the least number of the branch, which exists at time delay τ and k be the largest number, i.e., 3,

$$\tau_{\max}(m,l) \approx \tau \quad \text{and} \quad \tau_{\min}(k,l) \approx \tau,$$

up to terms of order one, see Fig. 3. Taking into account Eqs. (8) and (9), we obtain

$$mT_{\max} \approx kT_{\min} \approx \tau,$$

where the approximation sign means that the equality is satisfied up to the order 1 terms (k and m are large). All branches with numbers $m<n<k$ exist at time delay τ (see Fig. 3). Hence, the number of overlapping branches can be estimated as follows:

$$N \approx k - m = k - k\frac{T_{\min}}{T_{\max}} = k\frac{T_{\max} - T_{\min}}{T_{\max}} = \kappa_1\tau, \tag{10}$$

where the coefficient for this growth is given by

$$\kappa_1 = \frac{T_{\max} - T_{\min}}{T_{\max}T_{\min}} = \frac{1}{T_{\min}} - \frac{1}{T_{\max}}. \tag{11}$$

Expression (10) gives also a lower estimation for the number of coexisting periodic solutions of system (1). Indeed, if the branches are folded like in Fig. 2(a) then one branch may lead to more than one periodic solution for some delay values.

In a similar way, one can show that the number of coexisting branches in case 2 can be estimated as

$$N \approx \kappa_2\tau = \frac{1}{T_{\min}}\tau. \tag{12}$$

Finally note that there may exist few primary branches, which cannot be mapped one onto another by the transformation [Eq. (4)]. In this case, each branch will reappear with the increasing delay. The growth rate κ in this case is given as a superposition of the corresponding rates.

3. Turning points on branches

The branches may have turning points, which correspond to a fold bifurcation for the family of periodic solutions. The condition for the branch n to have a turning point can be written as follows:

$$\frac{\partial \tau(n,l)}{\partial l} = 0,$$

or taking into account Eq. (5),

$$1 + nT'(l) = 0. \tag{13}$$

Equation (13) can be rewritten as

$$T'(l) = -\frac{1}{n}. \tag{14}$$

With the increasing of branch number n, the fold point tends to some asymptotic value, which is independent on n and given by the condition $T'(l)=0$.

D. Example: Stuart-Landau oscillator with delay

In this paragraph we consider the following system:

$$z'(t) = (\alpha + i\beta)z(t) - z(t)|z(t)|^2 + z(t-\tau), \tag{15}$$

with the instantaneous part as the normal form for an oscillator close to the supercritical Andronov-Hopf bifurcation. Such system is called sometimes as the Stuart-Landau oscillator [3,45]. The additional term $z(t-\tau)$ accounts for a delayed feedback. Here $z(t)$ is a complex variable, i.e., the system has essentially two components, which can be chosen as real and imaginary parts of $z(t)$.

Due to symmetry properties of this system, some of its periodic solutions can be found analytically in the form of rotating waves $re^{i\omega t}$. Substituting this rotating wave into Eq. (15), we obtain the equation

$$i\omega = (\alpha + i\beta) - r^2 + e^{-i\omega\tau},$$

which leads to the following expressions for the amplitude r and frequency ω:

$$r = \sqrt{\alpha + \cos\omega\tau}, \tag{16}$$

$$\omega = \beta - \sin\omega\tau. \tag{17}$$

For the purposes of this paper, let us rewrite Eqs. (16) and (17) in terms of the amplitude r and the period $T=2\pi/\omega$:

$$r = \sqrt{\alpha + \cos\left(2\pi\frac{\tau}{T}\right)}, \tag{18}$$

$$T = \frac{2\pi}{\beta - \sin\left(2\pi\frac{\tau}{T}\right)}. \tag{19}$$

Expressions (18) and (19) are invariant under the change $\tau \to \tau + nT$, which reflects the fact that solutions on different branches are identical.

Although the period $T(\tau)$ along the branch is given in the implicit way by Eq. (19), one can obtain an explicit parametric representation of the branches with respect to T and τ. For this, we introduce an additional parameter $\psi=2\pi\tau/T$. With the help of this parameter, solutions of Eq. (19) can be written as follows:

$$T(\psi) = \frac{2\pi}{\beta - \sin\psi}, \tag{20}$$

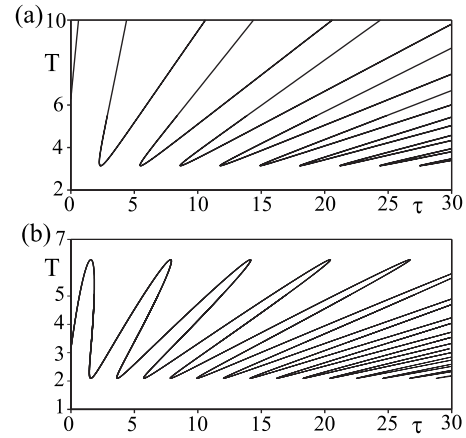

FIG. 4. Branches of periodic solutions of the Stuart-Landau oscillator with delay τ. T is the period along the branch. Parametric plot of Eqs. (20) and (21); (a) $\beta=1$; (b) $\beta=2$. The branches are independent on α provided $\alpha>1$. $0\le\psi\le 2\pi$.

$$\tau(\psi) = \frac{\psi + 2\pi n}{\beta - \sin\psi}. \tag{21}$$

Now the branches can be easily plotted by varying parameter ψ. For $\beta=1$ the branches are unbounded [see Fig. 4(a)], and for $\beta=2$ they are bounded [see Fig. 4(b)]. Moreover, for $\beta=2$ neighboring branches are connected. In both cases branches are independent on α provided $\alpha>1$ [see Eq. (19)].

As it is expected, the coexistence of multiple periodic solutions grows with the increasing of delay. The number of coexisting branches can be estimated using Eqs. (10) and (12) as

$$N \approx \frac{1}{T_{\min}}\tau = \frac{\beta+1}{2\pi}\tau$$

in the case $0<\beta<1$ and

$$N \approx \left(\frac{1}{T_{\min}} - \frac{1}{T_{\max}}\right)\tau = \frac{\tau}{\pi}$$

for $\beta>1$. Taking into account folding of the branches, the number of periodic solutions grows twice as fast, with the increasing of the delay, i.e., with the rates $(\beta+1)\tau/\pi$ and $2\tau/\pi$, respectively.

III. STABILITY OF THE REAPPEARING PERIODIC SOLUTIONS, LONG DELAY ISSUES

A. Some elements of the stability theory for periodic solutions

Let us introduce necessary notations and shortly remind basic elements of the stability theory [46] for periodic solutions of Eq. (1).

The linearization of Eq. (1) around some periodic solution $p(t)$ with a period T has the following form

$$\xi'(t) = A(t)\xi(t) + B(t)\xi(t-\tau), \tag{22}$$

where $A(t)=D_1 f[p(t),p(t-\tau)]$ and $B(t)=D_2 f[p(t),p(t-\tau)]$ are the T-periodic matrices $n\times n$. Here D_1 and D_2 denote partial derivatives with respect to the first and the second

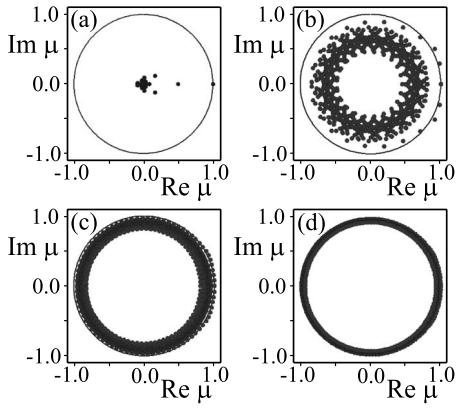

FIG. 5. Largest characteristic multipliers of a periodic solution of the delayed Duffing oscillator for different values of delay. Different figures correspond to the same solution on different branches: (a) branch $n=2(\tau=2.2)$, (b) $n=20(\tau=23.7)$, (c) $n=80(\tau=91.6)$, and (d) $n=140(\tau=159.5)$. Other parameters are $a=0.5$, $b=0.6$, and $d=0.06$.

arguments, respectively. Any solution of Eq. (22) with some initial function $q(t)$ can be represented as $x(t;s,q)=\Psi(t;s)q$, where $\Psi(t;s)$ is the evolution operator [46]. The monodromy operator is introduced as the evolution operator evaluated at the period;

$$U=\Psi(T;0).$$

Stability of the periodic solution $p(t)$ is determined by a countable set of *characteristic multipliers* [46,47] μ_j, $j=1,2,\ldots$, which are defined by the spectrum of U. The corresponding characteristic exponents are given as $\lambda_j=\frac{1}{T}\ln\mu_j$. A periodic solution is asymptotically stable if all its multipliers have modulus less than one except for the trivial one, which always equals to one. Equivalently, real parts of all characteristic exponents must be negative except for the trivial one. A bifurcation occurs when a multiplier crosses unitary circle as a parameter change.

Practically, characteristic multipliers and stability of periodic solutions can be computed using DDE-BIFTOOL software [44] provided the time delay it not too large.

B. Stability of periodic solutions versus delay

Considering τ as the control parameter, stability properties of periodic solutions are changing as τ is varied. In general, it is a challenging problem to find their stability, especially for larger τ. Figure 5 shows largest characteristic multipliers (with the largest modulus) for a periodic solution of the delayed Duffing oscillator [Eq. (2)] for different delay times. Even though the solution is the same, its stability properties change as one moves from one branch to another. One can clearly observe that more and more multipliers come to a small neighborhood of the unitary circle. This makes the problem degenerate and numerically stiff, i.e., the increasing of delay requires increasing of numerical precision in order to determine stability of a solution.

In the following we propose an analytical technique, which overcomes the appeared difficulty and shows how to approximate the characteristic multipliers for larger delay values. In particular we show that the characteristic multipliers have similar properties to the properties of the eigenvalues of stationary states for systems with large delay [3,41,42].

The linearization of Eq. (1) around the solution $x_n(t;\tau)$ on nth branch [this solution coincides with $x_0(t;l)$] at time delay $\tau=l+nT(l)$ has the form

$$\xi'(t)=A(t;l)\xi(t)+B(t;l)\xi(t-\tau), \tag{23}$$

where

$$A(t;l)=D_1f[x_0(t;l),x_0(t-l;l)],$$

$$B(t;l)=D_2f[x_0(t;l),x_0(t-l;l)] \tag{24}$$

are the T-periodic matrices, which depend only on function f and a shape of the solution $x_0(t;l)$. It is important that $A(t;l)$ and $B(t;l)$ do not depend on the branch number n and the time delay τ, at which the system is considered. This allows us to study stability properties of periodic solutions asymptotically with $\tau\to\infty$. Namely, by increasing τ (or, equivalently, branch number n) we are actually jumping from one branch to another by keeping the relative position l within the branch fixed, see Fig. 1. As a result, we consider the same periodic solution $x_0(t;l)$, which exists for different infinitely increasing delay values, and we study stability properties of this solution as τ increases. In the following, we consider τ to be continuous parameter and then apply the obtained results to the countable set $\tau(n,l)=l+nT(l)$, with $n=0,1,2,\ldots$ of delay values.

C. Pseudocontinuous spectrum

Here we show that periodic solutions possess a family of characteristic multipliers, which have the following asymptotic representation:

$$|\mu(\omega)|=1+\frac{1}{\tau}\gamma(\omega)+\mathcal{O}\left(\frac{1}{\tau^2}\right), \tag{25}$$

with the increasing of delay. Here ω is a parameter along the family and γ is some real function. Taking into account the relation $\mu=e^{\lambda T}$ between characteristic exponents and multipliers, the real parts of characteristic exponents have the following asymptotic representation:

$$T\,\mathrm{Re}\,\lambda(\omega)=\ln[|\mu(\omega)|]\approx\frac{1}{\tau}\gamma(\omega)+\mathcal{O}\left(\frac{1}{\tau^2}\right).$$

Using the analogy to the spectrum of eigenvalues for stationary solutions [3,41,42], we will call such spectrum *pseudocontinuous*. Its main features are the following:

(1) Pseudocontinuous spectrum tends to the critical value $|\mu|=1$ as $\tau\to\infty$.

(2) Its stability is determined by the sign of the function $\gamma(\omega)$.

(3) For any finite τ, the parameter ω admits a discrete countable number of values. As $\tau\to\infty$, the spectrum tends to a continuous in the sense that the discrete parameter ω covers densely the whole interval $[0,2\pi]$.

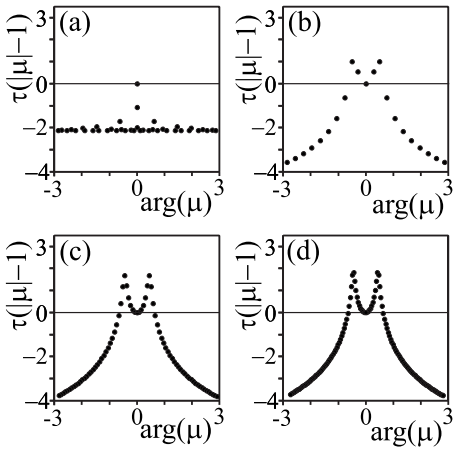

FIG. 6. Largest characteristic multipliers for a periodic solution of the Duffing oscillator with delay, which are rescaled accordingly to the rule $\gamma_j=\tau(|\mu_j|-1)$ (vertical axis) and $\omega=\arg(\mu_j)$ (horizontal axis). With the increasing of τ, such rescaled spectrum tends the continuous curve $\gamma(\omega)$ representing the pseudocontinuous spectrum as described by Eq. (25). Parameter values are as in Fig. 5.

In the case, if a periodic solution has only pseudocontinuous spectrum, its stability for large enough τ will be uniquely defined by the function $\gamma(\omega)$.

Before we give a rigorous proof for the existence of the pseudocontinuous spectrum, let us illustrate it using our numerical example of the Duffing oscillator [Eq. (2)]. Figure 6 shows approximations for the function $\gamma(\omega)$ by plotting $\gamma_j=\tau(|\mu_j|-1)$ versus $\omega=\arg(\mu_j)$, where μ_j are the numerically obtained largest characteristic multipliers. One can see that with increasing delay τ, the plot tends to some continuous curve, which determines stability of the periodic solution (it is unstable here). Below we give a proof for the existence of pseudocontinuous spectrum. Those readers, who are not interesting in details, may skip the rest in Sec. IV C.

Proof of the existence of pseudocontinuous spectrum. Here we use the theory for linear delay differential equations with periodic coefficients [46], which is analogous to the Floquet theory for ordinary differential equations. This theory implies that $\mu=e^{\lambda T}$ is the characteristic multiplier of Eq. (23) if and only if there is a nonzero solution of Eq. (23) of the form

$$\xi(t)=p(t)e^{\lambda t}, \tag{26}$$

where $p(t)=p(t+T)$ is periodic. Substituting Eq. (26) into Eq. (23), we obtain

$$p'(t)=(A(t;l)-\lambda\mathrm{Id})p(t)+e^{-\lambda\tau}B(t;l)p(t-\tau). \tag{27}$$

Since $p(t)$ is T-periodic, we have $p(t-\tau)=p[t-l-nT(l)]=p(t-l)$ and system (27) reduces to

$$p'(t)=[A(t;l)-\lambda\mathrm{Id}]p(t)+e^{-\lambda\tau}B(t;l)p(t-l). \tag{28}$$

where the large parameter τ occurs only as a parameter in $e^{-\lambda\tau}$. Thus, the corresponding monodromy operator of Eq. (28),

$$U=U(\lambda,e^{-\lambda\tau}),$$

depends only on λ and $e^{-\lambda\tau}$ smoothly. Since the linear system [Eq. (28)] possesses the periodic solution $p(t)$ by construction, one of its characteristic multipliers equals to one. This leads to the following condition on the monodromy operator:

$$[U(\lambda,e^{-\lambda\tau})-\mathrm{Id}]p=0,$$

which must hold for some periodic function $p(t)$. In general case, this is a codimension one condition [54], i.e., it leads to some characteristic equation,

$$F(\lambda,e^{-\lambda\tau})=0, \tag{29}$$

for determining the characteristic exponents λ. In order to construct this characteristic equation, one can use the determinant of the characteristic matrix introduced in [54] by formula (2.8).

Equation (29) allows proving the existence of pseudocontinuous spectrum. Indeed, substituting

$$\lambda=\frac{\gamma}{\tau T}+i\frac{\omega}{T} \tag{30}$$

into Eq. (29), we obtain up to the leading order

$$F\left(i\frac{\omega}{T},e^{-\gamma/T}e^{-i\omega/T\tau}\right)=0. \tag{31}$$

New unknown real variables in Eq. (31) are ω and γ instead of complex λ in Eq. (29). In the following we will proceed similarly to the case of the pseudocontinuous spectrum for stationary states [42]. Let us introduce the artificial parameter φ instead of rapidly growing phase $\frac{\omega}{T}\tau$,

$$F\left(i\frac{\omega}{T},e^{-\gamma/T}e^{-i\varphi}\right)=0. \tag{32}$$

The obtained extended Eq. (32) can be generically solved with respect to $\gamma(\omega)$ and $\varphi(\omega)$ since the equation is complex, i.e., it contains two real equations for two variables γ and φ. The obtained function $\gamma(\omega)$ is the resulting asymptotic function, which determines the pseudocontinuous spectrum (see Fig. 6). Coming back from the extended Eq. (32) to the original one [Eq. (31)], we additionally have to take into account the condition

$$\frac{\omega}{T}\tau=\varphi(\omega)+2\pi k,\quad k=0,\pm 1,\pm 2,\ldots$$

or equivalently

$$\frac{\omega}{T}=\frac{1}{\tau}\varphi(\omega)+\frac{2\pi k}{\tau}. \tag{33}$$

The last Eq. (33) determines the discrete values of $\omega=\omega_k$, which correspond to the discrete values of the pseudocontinuous spectrum $\gamma(\omega_k)$. As τ increases, the set of ω_k covers densely the whole domain of ω.

As a result, the pseudocontinuous spectrum approaches the continuous one as $\tau\to\infty$. Note, that the asymptotically continuous spectrum $\gamma(\omega)$ is determined by a regular system of Eqs. (32), which no longer contains the large parameter τ.

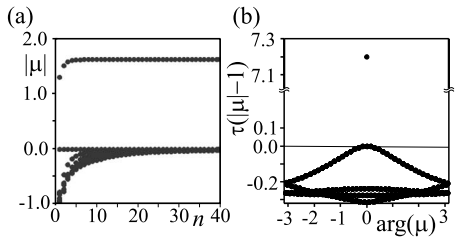

FIG. 7. Example of a strongly unstable spectrum for a periodic solution of Duffing oscillator with delay. Single multiplier with large modulus is the strongly unstable one. The others are belonging to the pseudocontinuous spectrum. (a) Modulus of largest characteristic multipliers versus branch numbers (delay is increasing). (b) Rescaled spectrum, i.e., $\tau(|\mu|-1)$ versus argument of μ for largest characteristic multipliers of a periodic solution on the branch 40. Note the scale on the vertical axis. Parameter values: $a=0.5$, $b=0.6$, and $d=0.06$.

Finally we remark that characteristic exponents [Eq. (30)] correspond to characteristic multipliers [Eq. (25)].

D. Strongly unstable spectrum

For completeness we show that another type of characteristic multipliers may appear, which have different asymptotics for large τ. These multipliers are not approaching the threshold value $|\mu|=1$ as $\tau\to\infty$ but tending to some unstable value

$$\mu \to \bar{\mu}, \quad |\bar{\mu}| > 1.$$

Since $|\bar{\mu}|>1$, the corresponding periodic solution is unstable. In the case when some periodic solution has a strongly unstable multiplier, close-by solutions will diverge on a time interval much shorter than the delay τ because the divergence rate is independent on the delay.

Strongly unstable spectrum may consist of finite number of multipliers (less than n) and it occurs if and only if the instantaneous part of Eq. (23), i.e., the system of ordinary differential equations (ODE),

$$\xi'(t) = A(t;l)\xi(t). \tag{34}$$

is unstable. The unstable Floquet multipliers of this system will serve as asymptotic values $\bar{\mu}$ for the strongly unstable characteristic multipliers of the system with delay (23). An example of strongly unstable spectrum is shown in Fig. 7. Why does strongly unstable spectrum exist?

The idea of the proof is the following. Assume that there is a characteristic multiplier μ with $|\mu(\tau)|>1$, which persists for all $\tau\to\infty$ and do not scale with τ. Then $\mu^{-\tau/T}\to 0$ with $\tau\to\infty$ and the delayed term

$$e^{-\lambda\tau}B(t;l)p(t-l) = \mu^{-\tau/T}B(t;l)p(t-l) \tag{35}$$

in Eq. (28) becomes exponentially small comparing with the term $[A(t;l)-\lambda\mathrm{Id}]p(t)$. In fact, choosing large enough τ, it can be made arbitrary small. Therefore, Eq. (28) can be formally reduced to the ODE

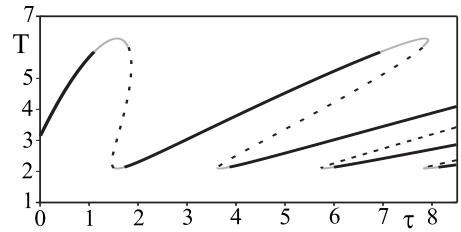

FIG. 8. This figure illustrates how branches of periodic solutions of the Stuart-Landau oscillator have asymptotically well-defined stability structure. Black solid lines ($-1.22<\psi \bmod 2\pi<1.18$) correspond to asymptotically stable parts of the branch for large enough delay, dashed ($1.868<\psi \bmod 2\pi<4.415$) are strongly unstable, and the remaining parts are weakly unstable and shown in gray. Parameter values are $\alpha=\beta=2$.

$$p'(t) = (A(t;l) - \lambda\mathrm{Id})p(t). \tag{36}$$

The condition for Eq. (36) to have a multiplier equal to one reduces to the equation

$$\det[U_0 - e^{\lambda T}\mathrm{Id}] = 0, \tag{37}$$

where U_0 is the monodromy matrix of the ODE [Eq. (34)]. Since the solutions of Eq. (37) are characteristic multipliers of Eq. (34), the strongly unstable multiplier will approach an unstable multiplier $\bar{\mu}=e^{\lambda T}$ of Eq. (34).

IV. ASYMPTOTIC STABILITY OF BRANCHES

Let us discuss the main consequences, which follow from the existence of the pseudocontinuous spectrum. First of all, let us note, that the sequence of periodic solutions $x_n(t,\tau)=x_0(t,l)$, which repeat themselves at time delays $l+nT(l)$, has a well defined stability limit as n increases. This means that all solutions from this sequence with large enough n will be stable if the corresponding pseudocontinuous spectrum is stable and no strongly unstable spectrum is present. Otherwise, the corresponding solutions will be unstable. In other words, in the limit of large delay, the branches of periodic solutions have well defined stability structure, i.e., there will be some stable part as well as unstable part. The unstable part can be again split into strongly unstable (if there are strongly unstable multipliers) or weakly unstable (when only pseudocontinuous spectrum is unstable). The corresponding parts can be described by the parameter l on the branches. Figure 8 illustrates this using the Stuart-Landau model with $\alpha=2$ and $\beta=2$, see caption to the figure.

Taking into account that almost all parts of the branches are eventually stretching with the increasing of delay, an increasing coexistence of stable as well as unstable periodic solutions is generally expected in systems with large delay. The relative fraction of stable solutions depends on a specific system, more exactly, on the asymptotic spectrum distribution along the branch.

V. COMPUTATION OF THE PSEUDOCONTINUOUS SPECTRUM

The main equation for finding branches $\gamma(\omega)$, to which the pseudocontinuous spectrum tends to, is given by Eq.

(32). As follows from Sec. IV, the equivalent problem can be formulated as follows:

For any given ω, find a value of $\gamma=\gamma(\omega)$ and $\varphi(\omega)$ such that the following linear system with delay

$$p'(t)=\left(A(t;l)-i\frac{\omega}{T}\mathrm{Id}\right)p(t)+e^{-\gamma/T-i\varphi}B(t;l)p(t-l) \quad (38)$$

has a multiplier 1. Here $l=\tau \bmod T$ and $A(t;l)$ and $B(t;l)$ are determined by linearizing Eq. (1) around the given periodic solution with period T; see Eq. (24). Equivalently, the following extended system can be considered:

$$x'(t)=f[x(t),x(t-\tau)],$$

$$p'(t)=\left\{D_1f[x(t),x(t-l)]-i\frac{\omega}{T}\mathrm{Id}\right\}p(t)+e^{-\gamma/T-i\varphi}D_2f[x(t),x(t-l)]p(t-l), \quad (39)$$

with the following additional conditions:

$$p(t)=p(t+T),$$

$$x(t)=x(t+T),$$

$$\|p(t)\|=1, \quad (40)$$

where the first two conditions ensure periodicity and the second one ensures that $p(t)$ is nontrivial. Additionally, one should add here a condition, which fixes the phase, e.g., $\mathrm{Im}\, p(0)=0$, since all functions of the form $p(t)e^{i\zeta}$ will be also solutions of Eq. (38). The obtained problem is a typical continuation problem, which no longer includes the large parameter τ. Standard continuation algorithms should be used in order to find the functions $\gamma(\omega)$ and $\varphi(\omega)$.

The implementation of the continuation algorithm will be discussed elsewhere [48]. Instead, we discuss here cases, for which the above problem can be significantly simplified.

Case 1. $\tau \bmod T=0$. This situation appears if system (1) has a periodic solution at $\tau=0$. In this case, Eq. (38) is reduced to the ODE,

$$p'(t)=\left[A(t;l)-i\frac{\omega}{T}\mathrm{Id}+e^{-\gamma/T-i\varphi}B(t;l)\right]p(t) \quad (41)$$

and the equivalent problem for finding functions $\gamma(\omega)$ and $\varphi(\omega)$ reduces to the finite-dimensional ODE continuation [49,50].

Case 2. System (1) has an additional phase shift symmetry. Examples of such systems are the Stuart-Landau oscillator [Eq. (15)] or the Lang-Kobayashi system [3,12,41,51–53]. In this case, the periodic orbits, which are invariant with respect to the symmetry, can be transformed into the stationary states by a suitable change of coordinated.

VI. CONCLUSIONS

To conclude, we have investigated properties of periodic solutions of systems with delay. In particular, we have shown that:

(i) Periodic solutions of systems with delay are organized in infinitely many branches.

(ii) The branches of periodic solutions can be obtained as the mapping [Eqs. (4) and (5)] of a primary branch on an appropriate interval of τ. From the practical points of view, it is sufficient to calculate only the primary branch.

(iii) The branches are eventually becoming wider with the increasing of τ, i.e., they occupy larger τ interval. As a result, the multistability of periodic solutions grows as delay increases.

(iv) This growth of the multistability is linear [Eq. (10)] and the corresponding estimation is given in Eqs. (11) and (12).

(v) One can effectively study asymptotic stability properties of periodic solutions as $\tau\to\infty$.

(vi) As τ becomes larger, the spectrum of characteristic multipliers of periodic solutions is generically split into two parts: pseudocontinuous spectrum and strongly unstable spectrum.

(vii) The main properties and implications of pseudocontinuous spectrum are explained. In particular, pseudocontinuous spectrum controls the destabilization of periodic solutions. It shows also that the destabilization mechanism of periodic solutions should be similar to that of spatially extended systems [1,3]. Moreover, it implies that the dimensionality of unstable manifolds of periodic orbits grows linearly as delay increases.

(viii) Strongly unstable spectrum is present when the instantaneous part of the linearization around the periodic solution is unstable. In this case, the feedback plays minor role, see also [23].

In our paper we have also outlined the scheme for practical computation of the pseudocontinuous spectrum although the numerical implementation of the corresponding path-following algorithm [see Eqs. (39) and (40)] will be discussed elsewhere [48].

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